

# Monoidal computer III: A coalgebraic view of computability and complexity

Dusko Pavlovic\*                      Muzamil Yahia†  
University of Hawaii, Honolulu, USA  
{dusko,muzamil}@hawaii.edu

## Abstract

Monoidal computer is a categorical model of *intensional* computation, where many different programs correspond to the same input-output behavior. The upshot of yet another model of computation is that a categorical formalism should provide a much needed high level language for theory of computation, flexible enough to allow abstracting away the low level implementation details when they are irrelevant, or taking them into account when they are genuinely needed. A salient feature of the approach through monoidal categories is the formal graphical language of *string diagrams*, which supports visual reasoning about programs and computations.

In the present paper, we provide a coalgebraic characterization of monoidal computer. It turns out that the availability of interpreters and specializers, that make a monoidal category into a monoidal computer, is equivalent with the existence of a *universal state space*, that carries a weakly final state machine for any pair of input and output types. Being able to program state machines in monoidal computers allows us to represent Turing machines, to capture their execution, count their steps, as well as, e.g., the memory cells that they use. The coalgebraic view of monoidal computer thus provides a convenient diagrammatic language for studying computability and complexity.

## 1 Introduction

In type theoretic semantics [6, 2, 24, II.3], an *extensional* model reduces computations to their set theoretic extensions, *computable functions*, whereas an *intensional* model also takes into account the multiple *programs* that describe

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each computable function.<sup>1</sup> From the outset, categorical semantics of computation adopted the extensional view, and interpreted universes of computation as cartesian closed categories of computable functions [20]. The idea of *monoidal computer* is to provide categorical semantics of intensional computation, by modifying the cartesian closed structure into a monoidal structure that should capture not only computable functions, but also their programs [32]. It turned out that the needed modification boiled down to two simple changes in the definition of the cartesian closed structure, which is displayed in the first line:

$$\frac{\mathcal{C}(X, [A, B]) \xrightleftharpoons[\cong]{\varepsilon_X^{AB}} \mathcal{C}(X \times A, B)}{\mathbb{C}^\natural(X, \mathbb{P}) \xrightarrow{\gamma_X^{AB}} \mathbb{C}(X \otimes A, B)} \quad (1)$$

The cartesian closed structure is thus defined by the bijections, indexed over the types  $A$  and  $B$ , and natural in  $X$ , between the morphisms  $X \times A \xrightarrow{f} B$  in  $\mathcal{C}$ , which we call *AB-applications*, and the corresponding *AB-function abstractions*  $X \xrightarrow{[f]} [A, B]$ . The **first change**, displayed in the second line, is that in a monoidal computer  $\mathbb{C}$ , the  $X$ -natural bijections  $\varepsilon_X^{AB}$  are relaxed to  $X$ -natural surjections  $\gamma_X^{AB}$ , which map  $X$ -indexed *programs*  $X \xrightarrow{F} \mathbb{P}$  to the corresponding *AB-applications*  $X \otimes A \xrightarrow{\{F\}} B$ . Each program thus evaluates to a single application, but many programs may evaluate to the same application.<sup>2</sup> The **second change** is that the *function types*  $[A, B]$ , representing the *AB-function abstractions*, must all be representable within a single *program type*  $\mathbb{P}$ .

It was shown in [32], and it will be summarized below, that the simple structure resulting from the step in (1) suffices for a succinct categorical reconstruction of the basic concepts of computability. But to capture the basic concepts of complexity<sup>3</sup>, we need, e.g., to count the operations executed in a computation, or the memory cells used. The high level view of computations as morphisms  $X \otimes A \rightarrow B$  in a monoidal category just displays the input type  $A$  and the output type  $B$ , plus a parameter type  $X$ , but does not seem to tell anything about the dynamics of computation. The attempt to extend the monoidal structure by a *grading* [33], capturing dynamics and complexity of computation, remained impractical.

In the present paper, we propose a *coalgebraic* method to capture dynamics of computation in vanilla monoidal computer, without any extensions. Coalgebra

<sup>1</sup>In semantics of programming languages, *denotational* models largely uphold the extensional view, whereas *operational* roughly correspond to the intensional view, although the parallel is imprecise, because the goals of semantics of programming languages are not identical to the goals of semantics of type theories.

<sup>2</sup>Remarkably, this is a genuine generalization only if the cartesian product  $X \times A$  in  $\mathcal{C}$  is replaced by a genuine tensor product  $X \otimes A$  in  $\mathbb{C}$ . If the tensor product happens to be cartesian, the surjections  $\gamma_X^{AB}$  split, and a cartesian closed structure can be extracted from the monoidal computer structure, under mild conditions.

<sup>3</sup>The original goal of the monoidal computer project was to provide a convenient framework for reasoning about computational hardness and logical depth in security applications, which seem beyond reach in the available low level models [29, 31].

has been, in a sense, an 'elephant in the room' of monoidal computer from the outset, since it is *the* method for capturing dynamic behaviors, which is our main goal here, and it has been used for that purpose even within the extensional framework of cartesian closed categories [19, 35, 43]. But noticing the obvious sometimes takes a while.

The conceptual coordinates of the approach are displayed in Table 1. The

Table 1: Semantical directions

extensional models: cartesian closed	$  \begin{array}{ccc}  [A, B] \times A & \xrightarrow{\varepsilon} & B \\  \nwarrow [f] \times A & & \nearrow f \\  & X \times A &  \end{array}  $ <p>abstractions <math>\leftrightarrow</math> applications</p>	$  \begin{array}{ccc}  [A^+, B] \times B & & \\  \nearrow \xi & \nwarrow \llbracket m \rrbracket \times B & \\  [A^+, B] \times A & & X \times B \\  \nwarrow \llbracket m \rrbracket \times A & \nearrow m & \\  & X \times A &  \end{array}  $ <p>behaviors <math>\leftarrow</math> machines</p>
intensional models: monoidal computers	$  \begin{array}{ccc}  \mathbb{P} \otimes A & \xrightarrow{\{\}} & B \\  \nwarrow \exists F \times A & & \nearrow \forall f \\  & X \otimes A &  \end{array}  $ <p>programs <math>\rightarrow</math> computations</p>	$  \begin{array}{ccc}  \mathbb{P} \otimes B & & \\  \nearrow \{\} & \nwarrow \exists M \otimes B & \\  \mathbb{P} \otimes A & & X \otimes B \\  \nwarrow \exists M \otimes A & \nearrow \forall m & \\  & X \otimes A &  \end{array}  $ <p>adaptive programs <math>\rightrightarrows</math> processes</p>

left column unpacks the correspondences in (1). The top row illustrates the way in which the application  $\leftrightarrow$  abstraction correspondence from the first row of (1), which defines the cartesian closed structure, readily lifts to the machine  $\rightarrow$  behavior correspondence of coalgebras and the induced final coalgebra homomorphisms. *Mutatis mutandis*, this latter correspondence, readily lifts to monoidal computers. This lifting is illustrated in the second row of the table, and is the subject of the present paper.

Each of the two rows displays a pair of correspondences which are under certain conditions equivalent with each other. For the first row, this equivalence boils down to the fact that a distributive category with free monoids (i.e., the list type constructors) is cartesian closed if and only if each category of coalgebras  $X \times A \rightarrow X \times B$  for a fixed  $A$  and  $B$  has a final object. The correspondence of applications and abstractions is thus equivalent with the correspondence of machines and their behaviors. The equivalence between the correspondences in the second row is the content of Thm. 4.3 below.

*Terminology.* A diligent reader may object that we sloppily use the term 'coalgebra' for morphisms in the form  $X \times A \rightarrow X \times B$ , while most coalgebraists

require that  $X$  is alone on the left. We have two reasons for this terminological abuse. The weaker reason is that coalgebras in the form  $X \rightarrow [A, X \times B]$  bijectively correspond to the morphisms  $X \times A \rightarrow X \times B$ , which are usually thought of as *Mealy machines* [12, 5]; and that the homomorphisms are in both cases the same. So the corresponding categories of machines and of coalgebras are isomorphic. The only difference is that the strict coalgebraic representation requires the closed structure, which is often not available. The stronger reason for calling machines coalgebras is that the machine models were studied coalgebraically long before the term ‘coalgebra’ was in use; and that the coalgebraic methods in general are much older than their categorical formalizations, as argued in [39, 27, 36, 37, 34]. Moreover, the coalgebraic methods are in practice often deeply interconnected with their algebraic counterparts [35, 34, 38]. Arguably, the essence of the coalgebraic methods is not that  $X$  is alone on the left of some arrow, but that they capture stateful behaviors as the *final* representations of machines and processes, whichever way they may be presented.

### Related work

While computability and complexity theorists seldom felt a need to learn about categories, there is a rich tradition of categorical research in computability theory, starting from one of the founders of category theory and his students [10, 25], through extensive categorical investigations of realizability [14, 13], to the recent work on Turing categories [7], and on a monoidal structure of Turing machines [3]. This recent work has, of course, interesting correlations with basic monoidal computer, but also substantial differences, arising from the different goals. We are not aware of any efforts towards a categorical semantics of computational complexity, but [1] seems to be close in spirit.

### Overview of the paper

In Sec. 2, we cover some preliminaries. In Sec. 3 we define monoidal computer, and state the Fundamental Theorem of Computability. In Sec. 4 we characterize monoidal computers coalgebraically. In Sections 5 and 6, we use this characterization to open a path towards studying computational complexity in monoidal computer. Sec. 7 offers some final comments.

## 2 Preliminaries

A monoidal computer is a *symmetric monoidal category* with some additional structure. As a matter of convenience, and with no loss of generality, we assume that it is a *strict* monoidal category. The reader familiar with these concepts may wish to skip to the next section. For the casual reader unfamiliar with these concepts, we attempt to provide enough intuitions to understand the presented ideas. The reader interested to learn more about monoidal categories should consult one of many textbooks, e.g. [21, VII.1,XI].

## 2.1 Monoidal categories

Intuitively, a monoidal category is a category  $\mathbb{C}$  together with a functorial monoid structure  $\mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C} \xleftarrow{I} \mathbb{1}$ . When  $\mathbb{C}$  is a monoidal *computer*, then we think of its objects  $A, B, \dots \in |\mathbb{C}|$  as datatypes, and of its morphisms,  $f, g, \dots \in \mathbb{C}(A, B)$  as computations. The tensor product  $A \otimes P \xrightarrow{f \otimes t} B \otimes Q$  then captures the parallel composition of the computations  $A \xrightarrow{f} B$  and  $P \xrightarrow{t} Q$ , whereas the categorical composition  $A \xrightarrow{f;g} C$  is the sequential composition of  $A \xrightarrow{f} B$  and  $B \xrightarrow{g} C$ .

With no loss of generality, we assume that tensors are *strictly* associative and unitary, and thus treat the objects  $A \otimes (B \otimes C)$  and  $(A \otimes B) \otimes C$  as the same, and do not distinguish  $A \otimes I$  and  $I \otimes A$  from  $A$ . This allows us to elide many parentheses and natural coherences [16, 21, Sec. VII.2]. Note, however, that the isomorphisms  $A \otimes B \xrightarrow{\sim} B \otimes A$  cannot be eliminated without causing a degeneracy.

*Notation.* When no confusion seems likely, we write

- $AB$  instead of  $A \otimes B$
- $\mathbb{C}(X)$  instead of  $\mathbb{C}(I, X)$

We omit the typing superscripts whenever the types are clear from the context.


**String diagrams.** A salient feature of monoidal categories is that the algebraic laws of the monoidal structure correspond precisely and conveniently to the geometric laws of *string diagrams*, formalized in [16], but going back to [40]. See also [42] for a survey. A string diagram usually consists of polygons or ovals linked by strings. In a monoidal computer, the polygons represent computations, whereas the strings represent data types, or the channels through which the data of the corresponding types flow. String diagrams thus display the data flows through composite computations. The reason why string diagrams are convenient for this is that the two program operations that usually generate data flows, the sequential composition  $f;g$  and the parallel composition  $f \otimes t$ , precisely correspond to the two geometric operations that generate string diagrams: one is the operation of connecting the polygons  $A \xrightarrow{f} B$  and  $B \xrightarrow{g} C$  by the string  $B$ , whereas the other one puts the polygons  $A \xrightarrow{f} B$  and  $P \xrightarrow{t} Q$  next to each other without connecting them. The associativity of these geometric operations then imposes the associativity law on the corresponding operations on computations. The identity morphism  $\text{id}_A$ , as the unit of the sequential composition, can be viewed as the channel of type  $A$ , and can thus be presented as the string  $A$  itself, or as an "invisible polygon" freely moved along the string  $A$ . The unit type  $I$  can be similarly presented as an "invisible string", freely added and removed to string diagrams. The algebraic laws of the monoidal structure are thus captured by the geometric properties of the string diagrams. The string crossings correspond to the symmetries  $A \otimes B \xrightarrow{\sim} B \otimes A$ .

## 2.2 Data services

We call *data service* the monoidal structure that allows passing the data around. In computer programs and in mathematical formulas, the data are usually passed around using variables. They allow copying and propagating the data values where they are needed, or deleting them when they are not needed. The basic features of a variable are thus that it can be freely copied or deleted. The basic data services over a type  $A$  in a monoidal category  $\mathbb{C}$  are

- the *copying* operation  $A \xrightarrow{\delta} A \otimes A$ , and
- the *deleting* operation  $A \xrightarrow{!} I$ ,

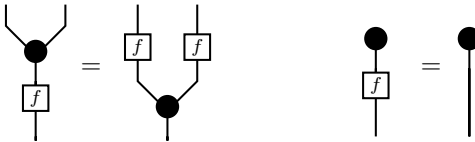
which together form a *comonoid*, i.e. satisfy the equations

$$\delta; (\delta \otimes A) = \delta; (A \otimes \delta) \qquad \delta; (! \otimes A) = \delta; (A \otimes !) = \text{id}_A$$


The correspondence between variables and comonoids was formalized and explained in [26, 30]. The associativity and the unit of the copying operation allow defining the unique  $n$ -ary copying operations  $A \xrightarrow{\delta} A^{\otimes n}$ , for all  $n \geq 0$ . The tensor products  $\otimes$  in  $\mathbb{C}$  are the cartesian products  $\times$  if and only if every  $A$  in  $\mathbb{C}$  carries a canonical comonoid  $A \times A \xleftarrow{\delta} A \xrightarrow{!} \mathbb{1}$ , where  $\mathbb{1}$  is the final object of  $\mathbb{C}$ , and all morphisms of  $\mathbb{C}$  are comonoid homomorphisms, or equivalently, the families  $A \xrightarrow{\delta} A \times A$  and  $A \xrightarrow{!} \mathbb{1}$  are natural. Cartesian categories are thus just the categories with natural families of copying and deleting operations.

**Definition 2.1.** A data service of type  $A$  in a monoidal category  $\mathbb{C}$  is a comonoid structure  $A \otimes A \xleftarrow{\delta} A \xrightarrow{!} I$ , where  $\delta$  provides the copying service, whereas  $!$  provides the deleting service. A data universe is a (strict) symmetric monoidal category  $\mathbb{C}$  with a chosen data service of each type  $A$ .

**Definition 2.2.** A morphism  $f \in \mathbb{C}(A, B)$  is a map if it is a comonoid homomorphism with respect to the data services on  $A$  and  $B$ , which means that it satisfies the following equations

$$f; \delta_B = \delta_A; (f \otimes f) \qquad f; !_B = !_A$$


For a data universe  $\mathbb{C}$ , we denote by  $\mathbb{C}^\natural$  the subcategory spanned by the maps with respect to its data services, i.e. by those  $\mathbb{C}$ -morphisms that preserve copying and deleting.

*Remark.* If  $\mathbb{C}$  is the category of relations, then the first equation says that  $f$  is a single-valued relation, whereas the second equation says that it is total. Hence the name.

Note that the morphisms  $\delta$  and  $!$  from the data services are maps with respect to the data service that they induce. They are thus contained in  $\mathbb{C}^\natural$ , and each of them forms a natural transformation with respect to the maps. This just means that the tensor  $\otimes$ , restricted to  $\mathbb{C}^\natural$ , is the cartesian product.

### 3 Monoidal computer

#### 3.1 Evaluation and evaluators

**Definition 3.1.** A monoidal computer is a data universe  $\mathbb{C}$  (as in Def. 2.1) with a distinguished type of programs  $\mathbb{P}$ , such that for every pair of types  $A, B$  there is a program evaluation, i.e. an  $X$ -natural family of surjections

$$\mathbb{C}^\natural(X, \mathbb{P}) \xrightarrow{\gamma_X^{AB}} \mathbb{C}(XA, B)$$

The next proposition says that the program evaluations from Def. 3.1 are a categorical view of Turing's *universal computer* [44], or of Kleene's *acceptable enumerations* [41, 24, II.5], or of *interpreters* and *specializers* from programming language theory [15].

**Proposition 3.2.** Let  $\mathbb{C}$  be a symmetric monoidal category with data services. Then specifying the program evaluations  $\gamma_X^{AB} : \mathbb{C}^\natural(X, \mathbb{P}) \rightarrow \mathbb{C}(XA, B)$  that make  $\mathbb{C}$  into a monoidal computer as defined in 3.1, is equivalent to giving for any three types  $A, B, C \in |\mathbb{C}|$  the following two morphisms:

- (a) a universal evaluator  $\{\}^{AB} \in \mathbb{C}(\mathbb{P}A, B)$  such that for every computation  $f \in \mathbb{C}(A, B)$  there is a program  $F \in \mathbb{C}^\natural(\mathbb{P})$  with

$$f(a) = \{\!F\!\} a$$

- (b) a partial evaluator  $[\!]^{(AB)C} \in \mathbb{C}^\natural(\mathbb{P}A, \mathbb{P})$  such that

$$\{\!G\!\}(a, b) = \{\![G, a]\!\} b$$

*Remark.* Note that the partial evaluators  $[]$  are maps, i.e. total and single valued morphisms in  $\mathbb{C}^\natural$ , whereas the universal evaluators  $\{\}$  are ordinary morphisms in  $\mathbb{C}$ . A recursion theorist will recognize the universal evaluators as Turing's *universal machines* [44], and the partial evaluators as Gödel's primitive recursive *substitution function*  $S$ , enshrined in Kleene's  $S_n^m$ -theorem [17]. A programmer can think of the universal evaluators as *interpreters*, and of the partial evaluators as *specializers* [15]<sup>4</sup>. In any case, (a) can be understood as saying that every computation can be programmed; and then (b) says that any program with several inputs can be evaluated on any of its inputs, and reduced to a program that waits for the remaining inputs:

$$g(a, b) = \{G\}(a, b) = \{[G, a]\} b$$

*Proof of Prop. 3.2.* Every natural transformation  $\gamma^{AB} : \mathbb{C}^\natural(-, \mathbb{P}) \rightarrow \mathbb{C}(- \otimes A, B)$  is uniquely determined by the computation  $\{\}^{AB} = \gamma_{\mathbb{P}}^{AB}(\text{id}_{\mathbb{P}}) \in \mathbb{C}(\mathbb{P} \otimes A, B)$ , because the naturality of  $\gamma^{AB}$  just means that

$$\gamma_X^{AB}(F) = \mathbb{C}(F, \mathbb{P}) \circ \gamma_{\mathbb{P}}^{AB}(\text{id}_{\mathbb{P}}) = (F \otimes A); \{\}^{AB} = \{F\}^{AB}$$

where we write  $\{F\}$  for  $(F \otimes A); \{\}$  not only for convenience, but also in reverence to Kleene's work and vision [18].

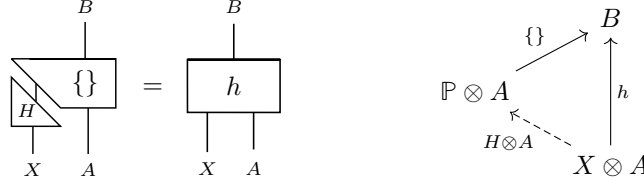
$$\begin{array}{ccc}
\mathbb{C}^\natural(\mathbb{P}, \mathbb{P}) & \xrightarrow{\gamma_{\mathbb{P}}^{AB}} & \mathbb{C}(\mathbb{P} \otimes A, B) \\
\downarrow \mathbb{C}^\natural(F, \mathbb{P}) & \begin{array}{ccc} \text{id}_{\mathbb{P}} & \mapsto & \{\} \\ \downarrow & & \downarrow \\ F & \mapsto & \{F\} \end{array} & \downarrow \mathbb{C}(F \otimes A, B) \\
\mathbb{C}^\natural(X, \mathbb{P}) & \xrightarrow{\gamma_X^{AB}} & \mathbb{C}(X \otimes A, B)
\end{array}$$

<sup>4</sup>When the theory is refined, it becomes useful to recognize subtle but important conceptual and technical distinctions.



Expressed in terms of the properties of the induced computation  $\{\}^{AB} \in \mathbb{C}(\mathbb{P} \otimes A, B)$ , the assumption that the program evaluations  $\gamma_X^{AB} : \mathbb{C}^\natural(X, \mathbb{P}) \rightarrow \mathbb{C}(X \otimes A, B)$  are surjective functions for all  $X$  means that for every computation  $h \in \mathbb{C}(X \otimes A, B)$  there is an  $X$ -indexed family of programs  $H \in \mathbb{C}^\natural(X, \mathbb{P})$  such that

$$\{H_x\} a = h(x, a) \quad (2)$$



By the Yoneda lemma [21, III.2], specifying a program evaluation  $\gamma^{AB} : \mathbb{C}^\natural(-, \mathbb{P}) \rightarrow \mathbb{C}(- \otimes A, B)$  is thus equivalent to specifying a computation  $\{\}^{AB} \in \mathbb{C}(\mathbb{P} \otimes A, B)$  satisfying (2). The task of proving the Proposition thus boils down to showing that  $(a) \iff (b) \Rightarrow (2)$ .

Setting  $X$  to be the tensor unit  $I$ , shows that  $\{\}^{AB} \in \mathbb{C}(\mathbb{P} \otimes A, B)$  induced as above by  $\gamma^{AB} : \mathbb{C}^\natural(-, \mathbb{P}) \rightarrow \mathbb{C}(- \otimes A, B)$  satisfies condition (a), and is thus a universal evaluator. To construct a partial evaluator  $[]^{ABC} \in \mathbb{C}^\natural(\mathbb{P} \otimes A, \mathbb{P})$  satisfying (b), consider the following diagram

$$\begin{array}{ccc}
 \mathbb{C}^\natural(\mathbb{P}, \mathbb{P}) & \xrightarrow{\gamma_{\mathbb{P}}^{BC}} & \mathbb{C}(\mathbb{P} \otimes B, C) \\
 \downarrow \mathbb{C}^\natural([],^{ABC}, \mathbb{P}) & \searrow \gamma_{\mathbb{P}}^{(AB)C} & \downarrow \mathbb{C}([],^{ABC} \otimes B, C) \\
 \mathbb{C}^\natural(\mathbb{P} \otimes A, \mathbb{P}) & \xrightarrow{\gamma_{\mathbb{P}A}^{BC}} & \mathbb{C}(\mathbb{P} \otimes A \otimes B, C)
 \end{array}$$

Since  $\gamma_{\mathbb{P}A}^{BC}$  is surjective, there must exist  $[]^{ABC} \in \mathbb{C}^\natural(\mathbb{P} \otimes A, \mathbb{P})$  such that

$$\gamma_{\mathbb{P}A}^{BC} ([]^{ABC}) = \gamma_{\mathbb{P}}^{(AB)C} (\text{id}_{\mathbb{P}})$$

Fix a choice of such  $[]^{ABC} \in \mathbb{C}^\natural(\mathbb{P} \otimes A, \mathbb{P})$ , and chase the above diagram. Recalling that  $\{\}^{(AB)C} = \gamma_{\mathbb{P}}^{(AB)C} (\text{id}_{\mathbb{P}})$  and  $\{\}^{BC} = \gamma_{\mathbb{P}}^{BC} (\text{id}_{\mathbb{P}})$ , and observing that  $\mathbb{C}^\natural([],^{ABC}, \mathbb{P}) (\text{id}_{\mathbb{P}}) = []^{ABC}; \text{id}_{\mathbb{P}} = []^{ABC}$ , the naturality of  $\gamma^{BC}$  implies that

$$\begin{aligned}
 ([]^{ABC} \otimes B); \{\}^{BC} &= \mathbb{C}([],^{ABC} \otimes B, C); \gamma_{\mathbb{P}}^{BC} (\text{id}_{\mathbb{P}}) = \\
 &= \gamma_{\mathbb{P}A}^{BC} \circ \mathbb{C}^\natural([],^{ABC}, \mathbb{P}) (\text{id}_{\mathbb{P}}) = \gamma_{\mathbb{P}}^{(AB)C} (\text{id}_{\mathbb{P}}) = \{\}^{(AB)C}
 \end{aligned}$$

Written in the bracket notation, this boils down to

$$\{[G, a]^{ABC}\}^{BC} b = \{G\}^{(AB)C} (a, b)$$

which shows that  $[]^{ABC}$  satisfies (b), as claimed.

Turning to the converse, suppose that universal evaluators  $\{\}^{AB}$  and partial evaluators  $[]^{ABC}$  are given, satisfying (a) and (b). We show that the universal evaluators  $\{\}^{AB}$  then satisfy the stronger requirement (2). Since we showed above that giving  $\{\}^{AB} \in \mathbb{C}(\mathbb{P} \otimes A, B)$  satisfying (2) is equivalent to specifying a natural family of surjections  $\gamma^{AB} : \mathbb{C}^\natural(-, \mathbb{P}) \rightarrow \mathbb{C}(- \otimes A, B)$ , this will complete the proof.

Towards the proof that  $(a) \wedge (b) \Rightarrow (2)$ , consider an arbitrary computation  $h \in \mathbb{C}(XA, B)$ . Then

- (a) gives  $\tilde{H} \in \mathbb{C}(\mathbb{P})$  such that  $\{\tilde{H}\}^{(XA)B}(x, a) = h(x, a)$ , and
- (b) gives  $H_x = [\tilde{H}, x]^{XAB} \in \mathbb{C}(X, \mathbb{P})$  such that  $\{H_x\}^{AB}a = h(x, a)$ .

$$h(x, a) = \{\tilde{H}\}(x, a) = \{[\tilde{H}, x]\}y$$

□

**Branching.** By extending the  $\lambda$ -calculus constructions as in [32], we can extract from  $\mathbb{P}$  the convenient types of natural numbers, truth values, etc. E.g., if the truth values  $\top$  and  $\perp$  are defined to be some programs for the two projections, then the role of the *if*-branching command can be played by the universal evaluator:

$$if(b, x, y) = \{b\}(x, y) = \begin{cases} x & \text{if } b = \top \\ y & \text{if } b = \perp \end{cases}$$

### 3.2 Examples of monoidal computer

The standard model of monoidal computer is the category **Cpf** of recursively enumerable sets and computable partial functions, or more precisely

- $|\mathbf{Cpf}| = \{A \subseteq \mathbb{N} \mid \exists e \in \mathbb{N}. \varphi_e(a) \downarrow \iff a \in A\}$
- $\mathbf{Cpf}(A, B) = \{f : A \multimap B \mid \exists e. \varphi_e = f\}$

Here the harpoon arrow in  $A \rightarrowtail B$  denotes the partial functions, whereas  $\{\varphi_e \mid e \in \mathbb{N}\}$  is an acceptable enumeration of computable partial functions [24, II.5]. The notation  $\varphi_e(a) \downarrow$  means that the partial function  $\varphi_e : \mathbb{N} \rightarrow \mathbb{N}$  is defined on  $a$ . The program type  $\mathbb{P}$  can be any language containing a Turing complete set of expressions. If the programs are encoded as natural numbers, then  $\mathbb{P} = \mathbb{N}$ ; if they are strings of bits, then  $\mathbb{P} = \{0, 1\}^*$ . The universal evaluators can be implemented, e.g., as a fixed family of universal Turing Machines. The partial evaluators are the total recursive functions constructed in Kleene's  $S_n^m$ -theorem [17]. Extending this model to recursive relations and nondeterministic Turing machines leads to the evaluators that obey the same laws, but introduces many nonstandard data services [28], which can be used to encode nonstandard algorithms [30]. A *quantum* monoidal computer can be defined within the category of complex Hilbert spaces, with all linear maps as morphisms. The data services are provided by Frobenius algebras [9]. The category of functions with respect to these data services is equivalent with the category of sets and functions [8], so the programs are still classical, and the evaluators can be defined as in [4]. It is important to note, however, that the program evaluations  $\gamma^{AB}$  are not surjective in a set-theoretic sense, but dense topologically, as spelled out in [4]. Lastly, let us mention that any *reflexive domain* [11] gives rise to an *extensional* monoidal computer. For more detail see [32, Sec. 4.1].

### 3.3 Encoding all types

**Proposition 3.3.** *Every type  $B$  in a monoidal computer is a retract of the type of programs  $\mathbb{P}$ . More precisely, for every type  $B \in |\mathbb{C}|$  there are computations*

$$B \begin{array}{c} \xrightarrow{e} \\ \xleftarrow{d} \end{array} \mathbb{P}$$

such that  $e^B$  is a map, and  $e^B; d^B = \text{id}_B$ . We often call  $e^B$  the encoding of  $B$  and  $d^B \in \mathbb{C}(\mathbb{P}, B)$  is the corresponding decoding.

*Remark.* Note that there is no claim that either  $e^B$  or  $d^B$  is unique. Indeed, in nondegenerate monoidal computers, each type  $B$  has many different encoding pairs  $e^B, d^B$ . However, once such a pair is chosen, the claim that  $e^B$  is a map means that it assigns a unique program code to each element of  $B$ . The fact that  $d^B$  is not a map means that some programs in  $\mathbb{P}$  may not decode to elements of  $B$ .

Since Prop. 3.3 says that the program evaluations make every type in a monoidal computer into a retract of  $\mathbb{P}$ , and Prop. 3.2 reduced the structure of monoidal computer to the evaluators for all types, it is natural to ask if the evaluators of all types can be reduced to the evaluators over the type  $\mathbb{P}$  of programs. Can all structure of a monoidal computer be derived from the structure of the type  $\mathbb{P}$  of programs? E.g., can the program evaluations be “uniformized” by always encoding the input data of all types in  $\mathbb{P}$ , performing the evaluations to get the outputs in  $\mathbb{P}$ , and then decoding the outputs back to

the originally given types? Can the type structure and the evaluation structure of a monoidal computer be reconstructed by unfolding the structure of  $\mathbb{P}$ , as it is the case in models of  $\lambda$ -calculus [20, I.15-I.18]. Is monoidal computer not yet another categorical view of partial applicative structures, or of  $\lambda$ -calculus?

The answer to all these question is positive *precisely* in the degenerate case of *extensional* monoidal computer. If the type structure of monoidal computer can be faithfully encoded in  $\mathbb{P}$ , then there is a retract of  $\mathbb{P}$  which supports an extensional model of computation, i.e. allows assigning a unique program to each computation.

If all evaluators can be derived by decoding the evaluators with the output type  $\mathbb{P}$ , and if the decoding preserves the original evaluators on  $\mathbb{P}$ , then all computation representable in monoidal computer must be provably total and single valued: it degenerates to a cartesian closed category derived from a  $C$ -monoid.

*Proof.* The claimed retraction  $B \begin{smallmatrix} \xrightarrow{e} \\ \xleftarrow{d} \end{smallmatrix} \mathbb{P}$  can be found using the following diagram:

$$\begin{array}{ccc}
 \mathbb{C}^{\natural}(\mathbb{P}, \mathbb{P}) & \xrightarrow{\gamma_{\mathbb{P}}^{I\mathbb{P}}} & \mathbb{C}(\mathbb{P}, B) \\
 \downarrow \mathbb{C}^{\natural}(e^B, \mathbb{P}) & \begin{array}{ccc} \text{id}_{\mathbb{P}} & \xrightarrow{\quad} & d^B \\ \downarrow & & \downarrow \\ e^B & \xrightarrow{\quad} & \text{id}_B \end{array} & \downarrow \mathbb{C}(e^B, B) \\
 \mathbb{C}^{\natural}(B, \mathbb{P}) & \xrightarrow{\gamma_B^{I\mathbb{P}}} & \mathbb{C}(B, B)
 \end{array}$$

While  $d^B$  is defined to be *the* image of  $\text{id}_{\mathbb{P}}$  along  $\gamma_{\mathbb{P}}^{I\mathbb{P}}$ ,  $e^B$  is defined to be *any* inverse image of  $\text{id}_B$  along  $\gamma_B^{I\mathbb{P}}$ , which must exist because  $\gamma_B^{I\mathbb{P}}$  is a surjection. so

$$\gamma_{\mathbb{P}}^{I\mathbb{P}}(\text{id}_{\mathbb{P}}) = d^B \quad \text{and} \quad \gamma_B^{I\mathbb{P}}(e^B) = \text{id}_B$$

The fact that  $e^B ; d^B = \text{id}_B$  follows from the naturality of  $\gamma^{I\mathbb{P}}$ , which implies that the square in the diagram commutes, and therefore

$$\begin{aligned}
 e^B ; d^B &= \mathbb{C}(e^B, B)(d^B) = \mathbb{C}(e^B, B) \circ \gamma_{\mathbb{P}}^{I\mathbb{P}}(\text{id}_{\mathbb{P}}) = \\
 &= \gamma_B^{I\mathbb{P}} \circ \mathbb{C}^{\natural}(e^B, \mathbb{P})(\text{id}_{\mathbb{P}}) = \gamma_B^{I\mathbb{P}}(e^B) = \text{id}_B
 \end{aligned}$$

□

*Remark.* In [32] we only considered the *basic* monoidal computer, where all types were the powers of  $\mathbb{P}$ . In the standard model, the programs are encoded as natural numbers, and all data are the tuples of natural numbers. Prop. 3.3 implies that all types must also be recursively enumerable in the internal sense of  $\mathbb{C}$ .

### 3.4 The Fundamental Theorem of Computability

In this section we show that every monoidal computer validates the claim of Kleene's fundamental result, which he called the Second Recursion Theorem [17, 22].

**Theorem 3.4.** *In every monoidal computer  $\mathbb{C}$ , for every computation  $g \in \mathbb{C}(\mathbb{P}A, B)$  there is a program  $\Gamma \in \mathbb{C}(\mathbb{P})$  such that*

$$g(\Gamma, a) = \{\Gamma\}a$$

We call  $\Gamma$  Kleene's fixed program of  $g$ .

*Proof.* Let  $G$  be a program such that

$$g([p, p], a) = \{G\}(p, a)$$

A Kleene fixed program  $\Gamma$  can now be constructed by evaluating  $G$  on itself, i.e. as  $\Gamma = [G, G]$ , because

$$g(\Gamma, a) = g([G, G], a) = \{G\}(G, a) = \{[G, G]\}a = \{\Gamma\}a$$

Applying and re-applying the Fundamental Theorem, it is not too hard to derive convenient representations of integers, arithmetic, primitive recursion,

and unbounded search, and thus derive a proof that monoidal computer is Turing complete. In [32], this was shown by interpreting the  $\lambda$ -calculus. In the next section, we provide yet another proof, by implementing Turing machines.

## 4 Coalgebraic view

So far, we formalized the *programs*  $\rightarrow$  *computations* correspondence from the left hand column of Table 1. But presenting computations in the form  $XA \xrightarrow{\{F\}} B$  only displays their interfaces, and hides the actual process of computation. To capture that, we switch to the right hand column of Table 1, and study the correspondence *adaptive programs*  $\rightarrow$  *processes*.

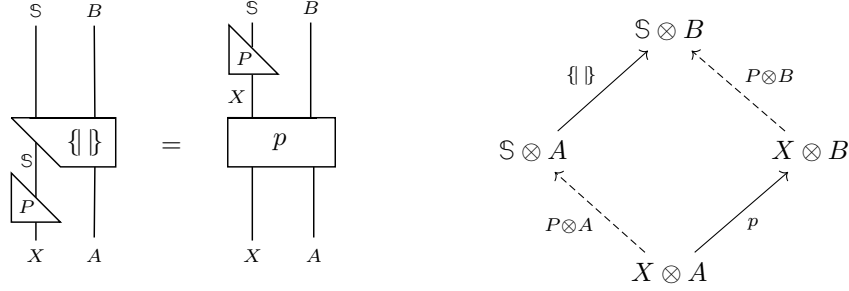
A *process* is presented as a morphism in the form  $X \otimes A \rightarrow X \otimes B$ . We interpreted the morphisms in the form  $X \otimes A \rightarrow B$  as  $X$ -indexed families of computations with the inputs from  $A$  and the outputs in  $B$ . The indices of type  $X$  can be thought of as the states of the world, determining which of the family of computations should be run. Interpreted along the same lines, a process  $X \otimes A \rightarrow X \otimes B$  does not only provide the output of type  $B$ , but it also updates the state in  $X$ . This is what state machines also do, and that is why the morphisms  $X \times A \xrightarrow{m} X \times B$  in cartesian categories are interpreted as machines. In a sufficiently complete cartesian category, such machines induce the coalgebra homomorphisms  $X \xrightarrow{[m]} [A^+, B]$ , which represent their *behaviors* in the final  $AB$ -machine  $[A^+, B] \times A \xrightarrow{\xi} [A^+, B] \times B$ , as in Table 1. A monoidal computer, though, turns out to provide a much stronger form of representation for its morphisms in the form  $X \otimes A \rightarrow X \otimes B$ : they are all represented in the same type  $\mathbb{P}$ , for all types  $A$  and  $B$ , although the representatives are not unique. This makes a fundamental conceptual difference, disinguishing mechanical processes from computational processes, which include life [23]. Every family of machines is designed in a suitable engineering language; but all computational processes can be programmed in any Turing complete language, just like all processes of life are programmed in the language of genes. That is why the morphisms  $X \otimes A \xrightarrow{p} X \otimes B$  are *processes*, and not merely machines. Their representations  $X \xrightarrow{P} \mathbb{P}$  are not merely  $X$ -indexed programs, but they are *adaptive* programs, since they adapt to the state changes, in the sense that we now describe.

**Definition 4.1.** A morphism  $XA \xrightarrow{p} XB$  in a monoidal category  $\mathbb{C}$  is an *AB-process*. If  $YA \xrightarrow{r} YB$  is another *AB-process*, then an *AB-process homomorphism* is a  $\mathbb{C}$ -morphism  $X \xrightarrow{f} Y$  such that  $(f \otimes A); r = p; (f \otimes B)$ . We denote by  $\mathbb{C}_{AB}$  the category of *AB-processes*.

**Definition 4.2.** A universal process in a monoidal category  $\mathbb{C}$  is carried by a universal state space  $\mathbb{S} \in |\mathbb{C}|$ , such that for every pair  $A, B \in |\mathbb{C}|$  there is a weakly final *AB-process*  $\mathbb{S}A \xrightarrow{\Downarrow^{AB}} \mathbb{S}B$ . The weak finality means that for every

process  $p \in \mathbb{C}(XA, XB)$  there is an  $X$ -adaptive program  $P \in \mathbb{C}^\natural(X, \mathbb{S})$  such that

$$\begin{aligned} \{P(x)\}_{\mathbb{S}} a &= P(p_X(x, a)) \\ \{P(x)\}_B a &= p_B(x, a) \end{aligned}$$



**Theorem 4.3.** Let  $\mathbb{C}$  be a symmetric monoidal category with data services. Then  $\mathbb{C}$  is a monoidal computer if and only if it has a universal process. The type  $\mathbb{P}$  of programs coincides with the universal state space  $\mathbb{S}$ .

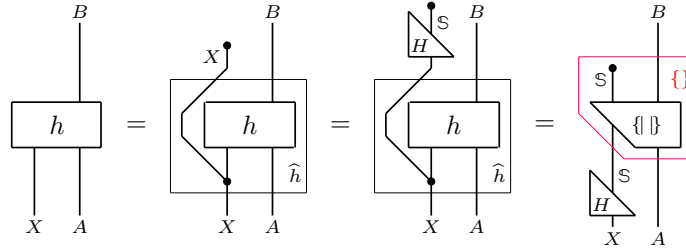
*Proof.* Given a weakly final  $AB$ -process  $\mathbb{S} \otimes A \xrightarrow{\{ \} \} \mathbb{S} \otimes B$ , we show that

$$\{ \}^{AB} = \left( \mathbb{S} \otimes A \xrightarrow{\{ \}^{AB}} \mathbb{S} \otimes B \xrightarrow{! \otimes B} B \right)$$

is a universal evaluator, and thus makes  $\mathbb{C}$  into a monoidal computer. Towards proving (2), suppose that we are given a computation  $X \otimes A \xrightarrow{h} B$ , and consider the process

$$\hat{h} = \left( X \otimes A \xrightarrow{\delta \otimes A} X \otimes X \otimes A \xrightarrow{X \otimes h} X \otimes B \right)$$

By Def. 4.2, there is then an  $X$ -adaptive program  $H \in \mathbb{C}^\natural(\mathbb{S})$  satisfying the rightmost equation in the next diagram.



The middle equation holds because  $H$  is in  $\mathbb{C}^\natural$ , i.e. a comonoid homomorphism. Deleting the state update from the process yields (2).

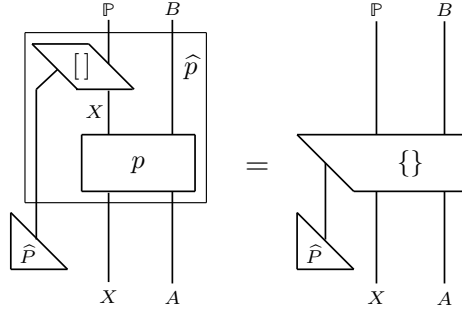
The other way around, if  $\mathbb{C}$  is a monoidal computer, with universal evaluators for all pairs of types, we claim that the weakly final  $AB$ -process is

$$\{ \}^{AB} = \left( \mathbb{P} \otimes A \xrightarrow{\{ \}^{A(\mathbb{P}B)}} \mathbb{P} \otimes B \right)$$

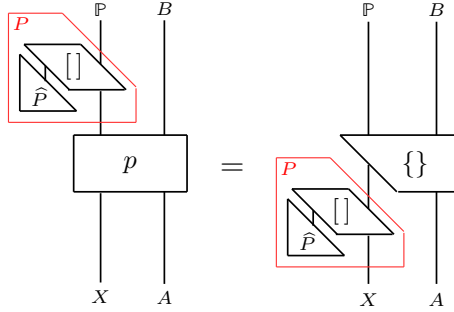
To prove the claim, take an arbitrary  $AB$ -process  $X \otimes A \xrightarrow{p} X \otimes B$ , and post-compose it with the partial evaluator on  $X$ , to get

$$\hat{p} = \left( \mathbb{P} \otimes X \otimes A \xrightarrow{\mathbb{P} \otimes p} \mathbb{P} \otimes X \otimes B \xrightarrow{[]^{XB\mathbb{P}} \otimes B} \mathbb{P} \otimes B \right)$$

Using the Fundamental Theorem of Computability, Thm. 3.4, construct a Kleene's fixed point  $\hat{P} \in \mathbb{C}(\mathbb{P})$  of  $\hat{p}$ .



The  $X$ -adaptive program  $P \in \mathbb{C}^\natural(\mathbb{P})$  corresponding to the process  $p \in \mathbb{C}(XA, XB)$  is now  $P(x) = [\hat{P}, x]^{XB\mathbb{P}}$ .



This completes the proof that  $\{\}^{A(B\mathbb{P})}$  satisfies definition 4.2 of weakly final  $AB$ -process, and that  $\mathbb{P}$  is thus not only a type of programs, but also a universal state space.  $\square$

## 5 Computability

In the remaining two sections we show how to run Turing machines in a monoidal computer, and how to measure their complexity. But a coalgebraic treatment of Turing machines as machines, in the sense discussed at the beginning of Sec. 4, would only display their behaviors, i.e. what rewrite and which move of the machine head will happen on which input, and it obliterates the configurations



of the tape, where the actual computation happens. In terms of Sec. 4, a Turing machine as a model of actual computation should not be viewed as a machine, but as a process. So we call them *Turing processes* here. While changing well established terminology is seldom a good idea, and we may very well regret this decision, the hope is that it will be a useful reminder that we are doing something unusual: relating Turing machines with adaptive programs, coalgebraically<sup>5</sup>. The presented constructions go through in an arbitrary monoidal computer, but require spelling out a suitable representation of the integers, and some arithmetic. This was done in [32], and can be done more directly; but for the sake of brevity, we work here with the category **Cpf** of recursively enumerable sets and computable partial functions from Sec. 3.2. The monoidal structure and the data services are induced by the cartesian products of sets, which are, of course not categorical products in a category of partial functions<sup>6</sup>. The monoidal category  $(\mathbb{C}, \otimes, I)$  will henceforth thus be  $(\mathbf{Cpf}, \otimes, \mathbb{1})$ .

Recall that Turing's definition of his machines can be recast<sup>7</sup> to present them as processes in the form

$$Q_\rho \otimes \Sigma \xrightarrow{\rho} Q_\rho \otimes \Sigma \otimes \Theta$$

where

- $Q_\rho$  is the finite set of states, always including the *final* state  $\checkmark \in Q_\rho$ ;
- $\Sigma$  is a fixed alphabet, the same for all  $\rho$ , always including the blank symbol  $\sqcup \in \Sigma$ ;
- $\Theta = \{\triangleleft, \square, \triangleright\}$  are the directions in which the head can move along the tape.

Let us recall the execution model: how these machines and processes compute. A Mealy machine  $Q_\kappa \times I \xrightarrow{\kappa} Q_\kappa \times O$  inputs a string  $n \xrightarrow{\iota} I$ , where  $n = \{0, 1, \dots, n-1\}$  sequentially, e.g. it reads the inputs  $\iota_0$ , then  $\iota_1$  etc, and it outputs a string  $n \xrightarrow{\omega} O$  in the same order, i.e.  $\omega_0, \omega_1$ , etc. In contrast, a Turing process in principle overwrites<sup>8</sup> its inputs, and outputs the results of overwriting when it halts; therefore, in a Turing process, the input alphabet  $I$  and its output alphabet  $O$  must be the same, say  $I = O = \Sigma$ . Both the inputs, and the outputs, and the intermediary data of a Turing process are in the form  $w : \mathbb{Z} \rightarrow \Sigma$ , where all but finitely many values  $w(z)$  must be  $\sqcup$ . So each word

<sup>5</sup>In view of [45], Turing would perhaps approve the link of computation and adaptation. In [23], von Neumann discussed it at length.

<sup>6</sup>The reason is that the singleton set, which is still the tensor unit, is not a terminal object for partial functions.

<sup>7</sup>This presentation was originally in [35], but it had to be moved into the Appendix, and the Appendix did not appear in the published version. The original version is available from first author's web page.

<sup>8</sup>There are models with a read-only input tape, and sometimes even with a write-only output tape. But then the inputs are copied on a working tape, where they are overwritten. The overwriting capability is essential, and must be realized in all models in one way or another.

$w : \mathbb{Z} \rightarrow \Sigma$  is still a finite string of symbols, like in the Mealy machine model; but  $w$  is written on the infinite 'tape', here represented by the set of integers  $\mathbb{Z}$ , along which the 'head' at each step either moves to the left, or it moves to the right, or it remains stationary. The position of the head is always 0, and the symbol that the head reads on that position is thus  $w(0)$ . If the process  $Q_\rho \otimes \Sigma \xrightarrow{\rho} Q_\rho \otimes \Sigma \otimes \Theta$ , which is a triple of functions  $\rho = \langle \rho_Q, \rho_\Sigma, \rho_\Theta \rangle$ , is defined on a given state  $q \in Q_\rho$  and a given input  $\sigma = w(0)$ , then it will

- overwrite  $\sigma$  with  $\sigma' = \rho_\Sigma(q, \sigma)$ ,
- transition to the state  $q' = \rho_Q(q, \sigma)$ , and
- move the head to the next cell in the direction  $\theta = \rho_\Theta(q, \sigma)$ .

If  $q = \checkmark$ , then  $\rho(\checkmark, \sigma) = \langle \checkmark, \sigma, \square \rangle$ , which means that the process must halt at the state  $\checkmark$ , if it ever reaches it.

To capture this execution model formally, we extend Turing processes over the alphabet  $\Sigma$ , first to processes over the set  $\tilde{\Sigma}$  of  $\Sigma$ -words written on a tape, and then to computations with the inputs and the outputs from  $\tilde{\Sigma}$

$$\frac{Q_\rho \otimes \Sigma \xrightarrow{\rho} Q_\rho \otimes \Sigma \otimes \Theta}{\frac{Q_\rho \otimes \tilde{\Sigma} \xrightarrow{\tilde{\rho}} Q_\rho \otimes \tilde{\Sigma}}{Q_\rho \otimes \tilde{\Sigma} \xrightarrow{\bar{\rho}} \tilde{\Sigma}}}$$

where

$$\tilde{\Sigma} = \left\{ w : \mathbb{Z} \rightarrow \Sigma \mid \text{supp}(w) < \infty \right\} \text{ where } \text{supp}(w) = \{z \mid w(z) \neq \sqcup\}$$

is the set of  $\Sigma$ -words written on a tape. The elements of  $\tilde{\Sigma}$  are often also called the *tape configurations*. Writing the tuples in the form  $\tilde{\rho} = \langle \tilde{\rho}_Q, \tilde{\rho}_{\tilde{\Sigma}} \rangle$ , define

$$\begin{aligned} \tilde{\rho}_Q(q, w) &= \rho_Q(q, w(0)) \\ \tilde{\rho}_{\tilde{\Sigma}}(q, w) &= w' \text{ where } w'(z) = \begin{cases} \tilde{w}(z-1) & \text{if } \rho_\Theta(q, w(0)) = \triangleleft \\ \tilde{w}(z) & \text{if } \rho_\Theta(q, w(0)) = \square \\ \tilde{w}(z+1) & \text{if } \rho_\Theta(q, w(0)) = \triangleright \end{cases} \text{ and} \\ \tilde{w}(z) &= \begin{cases} \rho_\Sigma(q, w(0)) & \text{if } z = 0 \\ w(z) & \text{otherwise} \end{cases} \end{aligned}$$

$$\bar{\rho}(q, w) = \begin{cases} w & \text{if } q = \checkmark \\ \bar{\rho}(\tilde{\rho}(q, w)) & \text{otherwise} \end{cases}$$

The execution of all Turing processes can now be captured as a single process

$$\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{\bar{\rho}} \mathbb{Q} \otimes \tilde{\Sigma}$$

where the state space  $\mathbb{Q}$  is the disjoint union of the state spaces  $Q_\rho$  of all Turing processes  $\rho \in \mathcal{T}$ , i.e.

$$\mathbb{Q} = \coprod_{\rho \in \mathcal{T}} Q_\rho \text{ where } \mathcal{T} = \{Q_\rho \otimes \Sigma \xrightarrow{p} Q_\rho \otimes \Sigma \otimes \Theta\}$$

so that the elements of  $\mathbb{Q}$  are the pairs  $\langle \rho, q \rangle$ , where  $q \in Q_\rho$ , and  $\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{p} \mathbb{Q} \otimes \tilde{\Sigma}$  is the pair  $p = \langle p_Q, p_{\tilde{\Sigma}} \rangle$  which, when applied to  $\langle \rho, q \rangle \in \mathbb{Q}$  and  $w \in \tilde{\Sigma}$ , gives:

$$p(\langle \rho, q \rangle, w) = \langle \langle \rho, q' \rangle, w' \rangle \text{ where } q' = \tilde{\rho}_Q(q, w) \text{ and } w' = \tilde{\rho}_{\tilde{\Sigma}}(q, w)$$

By applying Thm. 4.3 to the process  $\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{p} \mathbb{Q} \otimes \tilde{\Sigma}$ , we get the following

**Proposition 5.1.** *There is an adaptive program  $\tilde{P} \in \mathbf{Cpf}^{\sharp}(\mathbb{Q}, \mathbb{P})$  such that  $\tilde{P}(\rho, q)$  executes any Turing process  $\rho$  starting from the initial state  $q \in Q_\rho$ . This means that for every tape configuration  $w \in \tilde{\Sigma}$  holds*

$$\begin{aligned} \left\{ |\tilde{P}(\rho, q)| \right\}_{\mathbb{P}} w &= \tilde{P}(\rho, q') \\ \left\{ |\tilde{P}(\rho, q)| \right\}_{\tilde{\Sigma}} w &= w' \end{aligned}$$

where  $q' = \rho_Q(q, w(0))$  is the next state of  $\rho$ , and  $w' = \tilde{\rho}_{\tilde{\Sigma}}(q, w)$  is the next tape configuration. (The string diagram is the same as the one in Def. 4.2.)

**Corollary 5.2.** *The monoidal computer  $\mathbf{Cpf}$  is Turing complete.*

## 6 Complexity

### 6.1 Evaluating Turing processes

Using the process  $\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{p} \mathbb{Q} \otimes \tilde{\Sigma}$ , which according to Prop. 5.1 executes the single step transitions of Turing processes, we would now like to define a computation  $\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{\bar{p}} \tilde{\Sigma}$  that will evaluate Turing processes all the way; i.e. should execute all transitions that a process executes, and halt and deliver the output if the process halts, or diverge if the process diverges. The idea is to run something like the following pseudocode

$$\begin{aligned} \bar{p}(\langle \rho, q \rangle, w) &= \left( x := \langle \rho, q \rangle; y := w; \right. \\ &\quad \text{while } (p_Q(x, y) \neq \checkmark) \\ &\quad \quad \left\{ x := p_Q(x, y); y := p_{\tilde{\Sigma}}(x, y) \right\}; \\ &\quad \left. \text{print } y \right) \end{aligned} \tag{3}$$

We implement this program using the Fundamental Theorem of Computability. The function  $\bar{p}$  is derived as a Kleene's fixed program for an intermediary

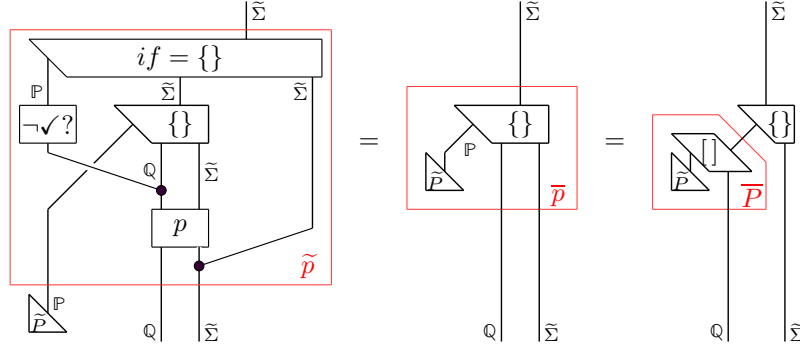
function  $\tilde{\rho}$ , lifting the derivation from Sec. 5.

$$\frac{\frac{Q \otimes \tilde{\Sigma} \xrightarrow{P} Q \otimes \tilde{\Sigma}}{P \otimes Q \otimes \tilde{\Sigma} \xrightarrow{\tilde{P}} \tilde{\Sigma}}}{Q \otimes \tilde{\Sigma} \xrightarrow{\tilde{P}} \tilde{\Sigma}}$$

The definition of  $\tilde{\rho}$  lifts the definition of  $\tilde{\rho}$  from Sec. 5, extended by an undetermined program  $\Upsilon$

$$\tilde{\rho}(\Upsilon, \langle \rho, q \rangle, w) = \begin{cases} w & \text{if } \rho_Q(q, w(0)) = \checkmark \\ \{\Upsilon\}(\langle \rho, q' \rangle, w') & \text{otherwise} \\ \quad \text{where } q' = \rho_Q(q, w(0)) \\ \quad \text{and } w' = p_{\tilde{\Sigma}}(\langle \rho, q \rangle, w) \end{cases}$$

Using the *if*-branching from Sec. 3.1, this schema can be expressed in a monoidal computer, as illustrated in the diagram below. Set  $\Upsilon$  to be Kleene's fixed program  $\tilde{P}$  of  $\tilde{\rho}$ , and define  $\bar{p} = \{\tilde{P}\}$ . This construction boils down to the first one of the following string diagram equations:



Given  $\langle \rho, q \rangle \in Q$  and  $w \in \tilde{\Sigma}$ ,  $\bar{p}$  thus runs  $\rho$  on  $w$ , starting from  $q$  and halting at  $\checkmark$ , at which point it outputs the current  $w$ . If it does not reach  $\checkmark$ , then  $\rho$  runs forever. The second equation in the above diagram proves the next proposition.

**Proposition 6.1.** *There is an adaptive program  $\bar{P} \in \text{Cpf}^{\dagger}(Q, \mathbb{P})$  that evaluates any Turing process  $\rho$  starting from a given initial state  $q \in Q_{\rho}$ . This means that for every tape configuration  $w \in \tilde{\Sigma}$  holds*

$$\{\bar{P}(\rho, q)\}w = \bar{\rho}(q, w)$$

## 6.2 Counting time

To count the steps in the executions of Turing processes, we add a counter  $i \in \mathbb{N}$  to the Turing process evaluator  $\bar{p}$ . The counter gets increased by at each execution step, and thus counts them. We call  $\bar{t}$  the computation which outputs

the final count. If  $\bar{p}$  halts, then  $\bar{t}$  outputs the value of the counter  $i$ ; if  $\bar{p}$  does not halt, then  $\bar{t}$  diverges as well. The pseudocode for  $\bar{t}$  could thus look something like this:

$$\begin{aligned} \bar{t}(\langle \rho, q \rangle, w) = & \left( x := \langle \rho, q \rangle; y := w; i := 0; \right. \\ & \textbf{while } (p_Q(x, y) \neq \checkmark) \\ & \quad \left\{ x := p_Q(x, y); y := p_{\tilde{\Sigma}}(x, y); i := i + 1 \right\}; \\ & \left. \textbf{print } i \right) \end{aligned} \quad (4)$$

The implementation of  $\bar{t}$  in a monoidal computer is similar to the implementation of  $\bar{p}$ . It follows a similar derivation pattern:

$$\frac{\frac{\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{p} \mathbb{Q} \otimes \tilde{\Sigma}}{\mathbb{P} \otimes \mathbb{Q} \otimes \tilde{\Sigma} \otimes \mathbb{N} \xrightarrow{\bar{t}} \mathbb{N}}}{\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{\bar{t}} \mathbb{N}}$$

where

$$\begin{aligned} \tilde{t}(\Upsilon, \langle \rho, q \rangle, w, i) &= \begin{cases} i & \text{if } \rho_Q(q, w(0)) = \checkmark \\ \{\Upsilon\}(\langle \rho, q' \rangle, w', i + 1) & \text{otherwise} \end{cases} \\ \bar{t}(\langle \rho, q \rangle, w) &= \{\tilde{T}\}(\langle \rho, q \rangle, w, 0) \end{aligned}$$

where

$$\begin{aligned} q' &= \rho_Q(q, w(0)) \\ w' &= p_{\tilde{\Sigma}}(\langle \rho, q \rangle, w) \end{aligned}$$

and  $\tilde{T}$  is Kleene's fixed program of  $\tilde{t}$ . It is easy to see, and prove, that  $\bar{t}(\langle \rho, q \rangle, w)$  halts if and only if  $\bar{p}(q, w)$  halts, and if it does halt, then it outputs the number of steps that  $\rho$  made before halting, having started from  $q$  and  $w$ . The string diagrams that implement  $\tilde{t}$ ,  $\tilde{T}$ ,  $\bar{t}$  and  $\bar{T}$  in a monoidal computer differ from the string diagrams that implemented  $\tilde{p}$ ,  $\tilde{P}$ ,  $\bar{p}$  and  $\bar{P}$  in the preceding section only by one additional string, for  $\mathbb{N}$ , with one additional operation on it, to increase the counter. This string gives, of course, the output of  $\bar{p}$ . Hence

**Proposition 6.2.** *There is an adaptive program  $\bar{T} \in \text{Cpf}^{\mathbb{h}}(\mathbb{Q}, \mathbb{P})$  that outputs the number of steps that a Turing process  $\rho$  makes in any run from a given initial state  $q \in Q_\rho$  to the halting state  $\checkmark$ . If the Turing process  $\rho$  starting from  $q$  diverges, then the computation  $\{\bar{T}(\rho, q)\}$  diverges as well. This means that, for every tape configuration  $w \in \tilde{\Sigma}$  holds*

$$\{\bar{T}(\rho, q)\}w = \bar{t}(\langle \rho, q \rangle, w)$$

### 6.3 Counting space

So far, we used the integers  $\mathbb{Z}$  as the index set for the tape configurations  $w : \mathbb{Z} \rightarrow \Sigma$ . The position of the head has always been  $0 \in \mathbb{Z}$ , and whenever the head moves, the tape configuration  $w$  gets updated to  $w' = \tilde{\rho}_{\Sigma}(q, w)$ , where  $w'(0)$  is the new position of the head, and the rest of the word  $w$  is reindexed accordingly, as described in Sec. 5. At each point of the computation  $w$  thus describes the tape content *relative to the current position of the head*; there is no record of the prior positions or contents.

To count the tape cells used by Turing processes, we must make the tape itself into a first class citizen. The simplest way to do this seems to be to add a counter  $m \in \mathbb{Z}$ , which denotes the offset of the current position of the head with respect to the initial position. This allows us to record how far up and down the tape, how far from its original position, does the head ever travel in either direction during the computation. To record these maximal offsets of the head, we need two more counters: let  $r \in \mathbb{Z}$  be the highest value that the head offset  $m$  ever takes; and let  $\ell \in \mathbb{Z}$  be the lowest value that the head offset  $m$  ever takes. The number of cells that the head has visited during the computation is then clearly  $r - \ell$ . To implement this space counting idea, we need to run a program roughly like this:

$$\begin{aligned} \bar{s}(\langle \rho, q \rangle, w) = & \left( x := \langle \rho, q \rangle; y := w; \ell, m, r := 0; \right. & (5) \\ & \text{while } (p_{\mathbb{Q}}(x, y) \neq \checkmark) \\ & \quad \left\{ \begin{array}{l} x := p_{\mathbb{Q}}(x, y); y := p_{\Sigma}(x, y); \\ \text{if } (\rho_{\Theta}(q, w(0)) = \triangleleft) \\ \quad \left\{ \text{if } (m = \ell) \{ \ell := \ell - 1 \}; m := m - 1 \right\} \\ \text{if } (\rho_{\Theta}(q, w(0)) = \triangleright) \\ \quad \left\{ \text{if } (m = r) \{ r := r + 1 \}; m := m + 1 \right\} \end{array} \right\} \\ & \left. \text{print } r - \ell \right) \end{aligned}$$

The derivation now becomes

$$\frac{\frac{\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{p} \mathbb{Q} \otimes \tilde{\Sigma}}{\mathbb{P} \otimes \mathbb{Q} \otimes \tilde{\Sigma} \otimes \mathbb{Z}^3 \xrightarrow{\tilde{s}} \mathbb{N}}}{\mathbb{Q} \otimes \tilde{\Sigma} \xrightarrow{\bar{s}} \mathbb{N}}$$

where

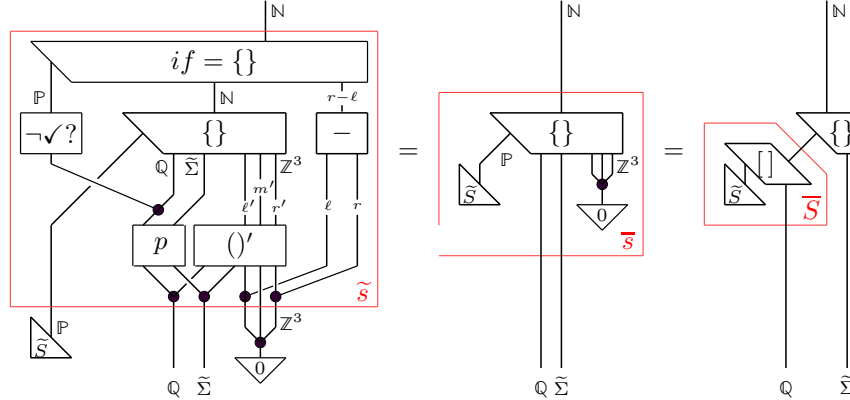
$$\tilde{s}(\Upsilon, \langle \rho, q \rangle, w, \ell, m, r) = \begin{cases} r - \ell & \text{if } \rho_Q(q, w(0)) = \checkmark \\ \{\Upsilon\}(\langle \rho, q' \rangle, w', \ell', m', r') & \text{otherwise} \end{cases}$$

$$\bar{s}(\langle \rho, q \rangle, w) = \{\tilde{S}\}(\langle \rho, q \rangle, w, 0, 0, 0)$$

where

$$\begin{aligned} q' &= \rho_Q(q, w(0)) \\ w' &= p_{\tilde{\Sigma}}(\langle \rho, q \rangle, w) \\ \ell' &= \begin{cases} \ell - 1 & \text{if } m = \ell \text{ and } \rho_{\Theta}(q, w(0)) = \triangleleft \\ \ell & \text{otherwise} \end{cases} \\ m' &= \begin{cases} m - 1 & \text{if } \rho_{\Theta}(q, w(0)) = \triangleleft \\ m & \text{if } \rho_{\Theta}(q, w(0)) = \square \\ m + 1 & \text{if } \rho_{\Theta}(q, w(0)) = \triangleright \end{cases} \\ r' &= \begin{cases} r + 1 & \text{if } m = r \text{ and } \rho_{\Theta}(q, w(0)) = \triangleright \\ r & \text{otherwise} \end{cases} \end{aligned}$$

and  $\tilde{S}$  is Kleene's fixed program of  $\tilde{s}$ . In a monoidal computer, the above constructions correspond to the following diagrams



The box  $()'$ , which computes  $\ell'$ ,  $m'$  and  $r'$  as above, is implemented by composing several branching commands, e.g. as described at the end of Sec. 3.1. Implementing this box is an easy but instructive exercise in programming monoidal computers. Put together, these constructions prove

**Proposition 6.3.** *There is an adaptive program  $\bar{S} \in \text{Cpf}^{\sharp}(\mathbb{Q}, \mathbb{P})$  that outputs the number of cells that a Turing process  $\rho$  uses in any run from a given initial*

state  $q \in Q_\rho$  to the halting state  $\checkmark$ . If the Turing process  $\rho$  starting from  $q$  diverges, then the computation  $\{\bar{S}(\rho, q)\}$  diverges as well. This means that, for every tape configuration  $w \in \tilde{\Sigma}$  holds

$$\{\bar{S}(\rho, q)\}w = \bar{s}(\langle \rho, q \rangle, w)$$

*Remark.* There are many variations of the above definitions in the literature, and several different counting conventions. E.g., an alternative to the above definition of  $\bar{s}$  would be something like

$$\bar{s}'(\langle \rho, q \rangle, w) = \{\tilde{S}\}(\langle \rho, q \rangle, w, w_\ell, 0, w_r)$$

where

$$\begin{aligned} w_\ell &= \min\{i \in \mathbb{Z} \mid w(i) \neq \sqcup\} \\ w_r &= \max\{i \in \mathbb{Z} \mid w(i) \neq \sqcup\} \end{aligned}$$

In contrast with  $\bar{s}$ , where the space counting convention is that a memory cell counts as used if and only if it is ever reached by the head, the space counting convention behind  $\bar{s}'$  is that every computation uses at least  $|w| = w_r - w_\ell$  cells, on which its initial input is written. If a Turing process halts without reading all of its input  $w$ , or even without reading any of it, the space used will still be  $|w|$ . Some textbooks adhere to the  $\bar{s}$ -counting convention, some to the  $\bar{s}'$ -counting convention, but many do not describe the process in enough detail to discern this difference. This is perhaps justified by the fact that the resulting complexity classes and their hierarchies are the same for all such subtly different counting conventions. E.g., the difference between  $\bar{s}$  and  $\bar{s}'$  is absorbed by the  $\mathcal{O}$ -notation, and only arises for computations that do not read their inputs.

## 7 Final comments

A bird's eye view of algebra and coalgebra in computer science suggests that algebra provides *denotational* semantics of computation, whereas coalgebra provides *operational* semantics [19, 35, 43]. Denotational semantics goes beyond the purely extensional view of computations (as maps from inputs to outputs), and models certain computational effects (such as non-termination, exceptions, non-determinism, etc.). Operational semantics goes further beyond the extensional view, and also modeling certain computational operations. Whereas the representations of computational effects of interest are thus generated by some suitable algebraic operations, computational behaviors are represented as elements of final coalgebras.

But although both the denotational and the operational approaches go beyond the purely *extensional* view, neither has supported a genuinely *intensional* view, where programs should be displayed as data in the universe of computations, as envisioned by Turing and von Neumann. Therefore, in spite



of the tremendous successes in understanding and systematizing computational structures and behaviors, categorical semantics of computation has remained largely disjoint from theories of computability and complexity.

The claim put forward in this paper is that coalgebra provides a natural, convenient, and powerful categorical setting for a fully intensional categorical theory of computability and complexity. The crucial step that enables this theory is the step from *final* coalgebras, that assign the *unique* descriptions to computational behaviors of some *fixed* types, to *universal* coalgebras, that assign *non-unique* descriptions to computations of *arbitrary* types. These non-unique descriptions of computations of arbitrary types by expressions of a single *universal* type are what we usually call *programs*.

The message of this paper is that *programmability is a coalgebraic property*, just like *computational behaviors are coalgebraic*. This message is formally expressed through *universal processes*; it can perhaps be expressed more generally through *universal coalgebras*, as families of weakly final coalgebras, all carried by the same *universal state space*. Theorem 4.3 spells out in the framework of monoidal computer the fact that every Turing complete programming language provides a universal coalgebra for computable functions of all types; and vice versa, every universal coalgebra induces a corresponding notion of program. Just like abstract computational behaviors of a given type are precisely the elements of a final coalgebra of that type, abstract programs are precisely the elements of a universal coalgebra. Just like final coalgebras can be used to define semantics of computational behavior [35], universal coalgebras can be used to define semantics of programs.

From a slightly different angle, the fact that universal coalgebras characterize monoidal computers, proven in Theorem 4.3, can also be viewed as a coalgebraic characterization of computability. There are, of course, many characterizations of computability. The upshot of this one is, however, in Propositions 6.2 and 6.3: the coalgebraic view of computability opens an alley towards complexity. In any universe of computable functions, normal complexity measures [33] can be programmed coalgebraically. Combining this coalgebraic view of complexity with the algebraic view of randomized computation seems to open up a path towards a categorical model of one-way functions, and towards categorical cryptography, which has been the original goal of this project.

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